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# Fuel Optimal Reorientation of Axisymmetric Spin-Stabilized Satellites


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## I. Introduction

THE problem of fuel optimal reorientation of axisymmetric spin-stabilized rigid satellites using reaction control jets was widely investigated in the 1960s. Adopting various approaches, researchers concluded that fuel optimal reorientation requires two impulsive torques when the maneuver time is large enough. However, such two-impulse schemes become invalid for small maneuver times. This Note applies a new numerical technique to solve the fuel optimal spin axis reorientation problem for large and small maneuver times.

For simplicity, early investigations of reorientation maneuvers employed impulsive torques with no implication of fuel optimality.[1] Later, the two-impulse scheme was shown to be fuel optimal for maneuver times that are large enough to accommodate the required precessional motion.[2] Next, Yin and Grimmel considered the bounded input problem, showed that the optimal control was bang off bang, and developed the associated switching curves. Furthermore, it was shown that reducing the maneuver time decreases the control off time until the fuel optimal control converges to the time optimal bang-bang control.[3] Later, the bounded fuel optimal problem was examined in detail by Wingate, who determined reorientation schemes for various geometries by numerically minimizing the associated Hamiltonian.[4] Recently, a new technique was developed for exactly solving unbounded fuel optimal control problems for arbitrary maneuver times.

Extending the work of Krasovskii,[5] Neustadt developed a geometric approach to solving optimal control problems.[6] This approach was applied to fuel optimal control and resulted in a control scheme that was demonstrated on second-order systems.[7] Recently, the development of an adaptive grid bisection search has made it possible to exactly solve fuel optimal control problems for higher order systems with a prescribed maneuver time.[8] This method is well suited for the rigid satellite problem, especially when rapid reorientation is desired.

The formulation of the fuel optimal reorientation of a spin-stabilized axisymmetric rigid satellite is presented in Sec. II. Nondimensional plots of fuel consumption vs maneuver time for various geometries are given in Sec. III, and some typical maneuvers are detailed. Finally, some concluding remarks are given in Sec. IV.

## II. Fuel Optimal Reorientation

A rigid satellite is shown in Fig. 1. The inertial coordinates $i_1$, $i_2$, and $i_3$ and the body-fixed coordinates $b_1$, $b_2$, and $b_3$ are related by successive rotations through an appropriate selection of Euler angles. The coordinate system is first rotated about the $i_3$ axis through the angle $\psi$. Next, the system is rotated about the nonspinning body $y$ axis through the angle $\theta$. Finally, the system is rotated about the body axis of symmetry $b_1$ thorugh the angle $\phi$. The satellite is spinning at the rate $\Omega$ about the axis of symmetry $b_1$ that has an associated mass moment of inertia $I_a$. The transverse mass moment of inertia is denoted by $I_t$. The satellite is reoriented by means of body-fixed reaction control jets producing bidirectional control moments about the $b_3$ axis. Assuming that the control moment is initially aligned with the nonspinning body $z$ axis, the linearized equations of motion are[4]

$$\frac{d^2\psi(\tau)}{d\tau^2} - r\frac{d\theta(\tau)}{d\tau} = \cos\tau u(\tau)$$

$$\frac{d^2\theta(\tau)}{d\tau^2} + r\frac{d\psi(\tau)}{d\tau} = \sin\tau u(\tau) \tag{1}$$

where $\tau = \Omega t$ is the nondimensional time, $u(\tau) = M(\tau)/I_t\Omega^2$ is the admissible nondimensional control moment, and $r = I_a/I_t$ is the inertia ratio. The equation of motion was linearized assuming that $\theta$ and $d\psi/d\tau$ are small. Note that $r$ ranges from 0 (long thin rod) to 2 (flat disk).

We define the state vector $x(\tau) = [\psi(\tau)\ \dot\psi(\tau)\ \theta(\tau)\ \dot\theta(\tau)]^T$ and convert Eq. (1) to the first-order form

$$\dot{x}(\tau) = Ax(\tau) + b(\tau)u(\tau) \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \\ 0 & -r & 0 & 0 \end{bmatrix} \tag{3a}$$

$$b(\tau) = \begin{bmatrix} 0 \\ \cos\tau \\ 0 \\ -\sin\tau \end{bmatrix} \tag{3b}$$

The objective of the control is to transfer the system from some initial state $x_0$ to a final state $x_1$ in maneuver time $\tau_f$ while minimizing the fuel

$$\text{fuel} = \int_0^{\tau_f} |u(\tau)|\, d\tau \tag{4}$$

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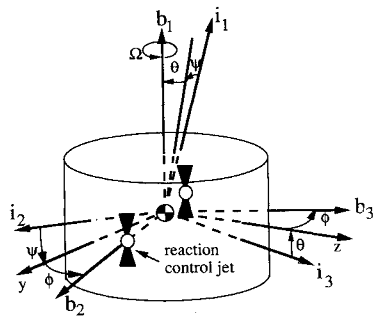


**Fig. 1 Satellite orientation angles.**

{Note in Eq. (4) the admissible control is any bounded or unbounded integrable function. For example, the unit impulse at time $\tau_0$ denoted $\delta(\tau-\tau_0)$ is an admissible control [$\delta(\tau-\tau_0)$ is said to have magnitude one since $\int_{-\infty}^{\infty}\delta(\tau-\tau_0)\,d\tau=1$].}

The solution to the initial value problem of Eq. (2) is

$$x(\tau)=e^{A\tau}\left(x_0+\int_0^{\tau}e^{-As}b(s)u(s)\,ds\right) \tag{5}$$

Rearranging Eq. (5), we define the reachable state as

$$y=\int_0^{\tau_f}e^{-A\tau}b(\tau)u(\tau)\,d\tau=e^{-A\tau_f}x_1-x_0 \tag{6}$$

To determine the fuel optimal control, we define the control determining function,

$$g(\eta,\tau)=\eta^T e^{-A\tau}b(\tau) \tag{7}$$

The normal vector $\eta$ is an element of the hyperplane $H=\{\eta:\eta^T y=1\}$. For the system of Eq. (2), $\eta=[\eta_1\ \eta_2\ \eta_3\ \eta_4]^T$ and the control determining function becomes

$$g(\eta,\tau)=\left(\frac{\eta_1}{r}-\eta_4\right)\sin(1-r)\tau+\left(\frac{\eta_3}{r}+\eta_2\right)\cos(1-r)\tau$$
$$-\frac{\eta_1}{r}\sin\tau-\frac{\eta_3}{r}\cos\tau \tag{8}$$

The fuel optimal control input has the form of a series of impulses expressed as[8]

$$u^*(\tau)=\frac{g^T c}{\alpha} \tag{9}$$

in which

$$\alpha=\min_{\eta\in H}\sup_{0\le\tau\le\tau_f}|g(\eta,\tau)| \tag{10}$$

$$g=\left\{\mathrm{sgn}\left[g(\eta,\tau_1)\right]\delta(\tau-\tau_1)\cdots\mathrm{sgn}\left[g(\eta,\tau_N)\right]\delta(\tau-\tau_N)\right\}^T \tag{11}$$

In Eq. (11), the optimal normal vector $\eta$ is an element of the hyperplane that satisfies Eq. (10), and $\tau_i\ (i=1,2,\ldots,N)$ represent nondimensional times at which $\alpha$ is realized. The impulse coefficient vector $c=[c_1\ c_2,\ldots,c_N]^T$ must satisfy

$$1=\sum_{r=1}^{N}c_r=\mathbf{1}^T c$$

and is chosen such that the control transfers the system to the desired final state. The optimal fuel consumed by the control given in Eq. (9) is equal to $1/\alpha$.

Solving Eq. (10) represents the primary difficulty associated with determining the fuel optimal control. For certain initial conditions and large enough $\tau_f$, it is possible to find a solution analytically. But as $\tau_f$ decreases, analytical solutions become difficult, if not impossible, to determine. Fortunately, a recently developed adaptive grid bisection search can be used to find a solution to Eq. (10) for large and small maneuver times.[8] In this technique, a portion of the hyperplane is discretized into a grid of normal vectors $\eta$. At each grid point,

$$\sup_{0\le\tau\le\tau_f}|g(\eta,\tau)|$$

is determined. These values are compared across the grid to determine which element most closely satisfies Eq. (10). This element is referred to as the grid optimal normal vector $\eta'$. The grid is then refined and centered about $\eta'$, and the process is repeated until the grid optimal normal vector converges to the optimal normal vector $\eta^*$. This process yields all of the necessary components of the optimal control except the impulse coefficients. These coefficients are then determined by substituting Eq. (9) into Eq. (6) and row reducing the resulting set of linear simultaneous equations.

## III. Reorientation Demonstration

The adaptive grid bisection search was used to determine the fuel optimal control that would transfer the system of Eq. (2) from the initial state $x_0=[0\ 0\ \theta_0\ 0]^T$ to the origin. Considering Eq. (6), the hyperplane is defined by $H=\{\eta:\eta_3=-1/\theta_0\}$. Nondimensional plots of optimal fuel consumption vs $\tau_f$ are given in Fig. 2. With the exception of $r=1$, each of the cases considered reaches its minimum possible fuel consumption over the range of maneuver times at $\tau_f=2\pi$. For $\tau_f\ge2\pi$, it is clear from Eq. (8) that the optimal normal vector $\eta^*=[0\ 1/r\theta\ -1/\theta\ 0]^T$, for all but the case when $r=1$. However, analytically determining the optimal normal vectors for these cases is difficult when $\tau_f<2\pi$. These solutions are determined using adaptive grid bisection. Note the sharp rise in fuel consumption as $\tau_f$ decreases.

For the case when $r=1$, the satellite behaves as a sphere. From Eq. (1), the period of the precessional motion is identical to the period of spin. Thus, the minimum possible fuel consumption is attained at $\tau_f=\pi$, the time required for the satellite to complete ½ revolution. For $\tau_f\ge\pi$, the optimal normal vector is $\eta^*=[0\ 1/r\theta\ -1/\theta\ \eta_4]^T$ where $\eta_4$ is free. Again, adaptive grid bisection reveals a sharp rise in fuel consumption for more rapid maneuvers.

Table 1 contains the optimal control parameters for a satellite with $r=1.5$ undergoing a 1-rad reorientation for four different maneuver times. The case with $\tau_f=4$ demonstrates a

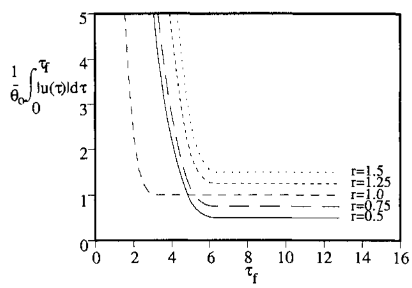


**Fig. 2 Minimum fuel vs control time.**

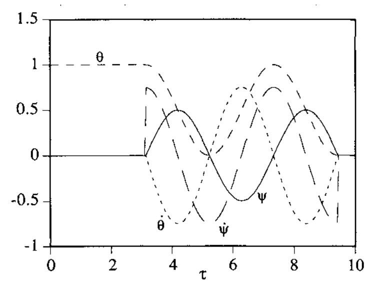

**Fig. 3 Reorientation maneuver with $r=1.5$ and $\tau_f=10$.**

**Table 1 Control parameters for $r = 1.5$**

| $\tau_f$ | Optimal fuel | $\tau_j$ | $\text{sgn}[g(\eta^*, \tau_j)]$ | $c_j$ |
|---|---|---|---|---|
| 4.0 | 6.5940863 | 0.0000000 | − | 0.1135749 |
| | | 1.0622951 | + | 0.2882963 |
| | | 2.9377049 | − | 0.3623254 |
| | | 4.0000000 | + | 0.2358035 |
| 6.0 | 1.5501346 | 0.2297864 | + | 0.4366778 |
| | | 3.3180872 | − | 0.1137695 |
| | | 6.0000000 | + | 0.4483121 |
| 8.0 | 1.5000000 | 0.0000000 | + | 0.5000000 |
| | | 3.1415927 | − | 0.0000000 |
| | | 6.2831853 | + | 0.5000000 |
| 10.0 | 1.5000000 | 0.0000000 | + | 0.0000000 |
| | | 3.1415927 | − | 0.5000000 |
| | | 6.2831853 | + | 0.0000000 |
| | | 9.4247780 | − | 0.5000000 |

rapid maneuver, which is characterized by high fuel consumption. This maneuver requires four impulses of different magnitudes. Three impulses of different magnitudes are required for $\tau_f = 6$. The optimal fuel required for this case is close to the minimum achieved for $\tau_f \geq 2\pi$. In contrast, only two impulses are required when $\tau_f = 8$, with the optimal fuel consumption equal to the minimum possible consumption. Further increases in $\tau_f$ offer no advantage in fuel consumption for this case. However, for large $\tau_f$ there may exist multiple solutions to the selection of impulse coefficients, enabling flexibility in the optimal control system design. For example, an infinite number of pulsing schemes are possible for the case $\tau_f = 10$. The coefficients contained in Table 1 were chosen such that the control requires only one-sided pulsing. This control is demonstrated in Fig. 3. A pulse applied at $\tau = \pi$ yields a step change in $\dot{\psi}$, exciting the precessional motion of the body. A second pulse applied at $\tau = 3\pi$ terminates the precessional motion and completes the desired reorientation maneuver.

## IV. Conclusion

A recently developed numerical technique for exactly solving fuel optimal control problems has been demonstrated on a classical problem. The adaptive grid bisection search was discussed and applied to the problem of fuel optimal reorientation of axisymmetric spin-stabilized rigid satellites. This approach was shown to be useful in determining the fuel optimal impulsive control strategy, particularly when rapid maneuvers are desired. For large enough maneuver times, the method produced the familiar two-impulse reorientation maneuver. For even larger maneuver times, the existence of multiple optimal pulsing schemes was revealed.


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# Ground Verification for Satellite High-Accuracy Onboard Antenna Drive Control Systems


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## Nomenclature

$F$ = suspension force of wire
$G$ = gravity center of rotation part [antenna pointing mechanism (APM) rotor and inertia dummy]
$K$ = rigidity of the APM rotation in gravity compensation
$k$ = rigidity of the APM bearings rotation
$N$ = drive torque of the actuator in the APM
$P$ = suspension point by wire
$T_{az}$ = torque around azimuth axis
$T_{el}$ = torque around elevation axis
$W$ = weight of the rotation part
$W_G$ = weight of the azimuth gimbal and the elevation gimbal
$Z$ = distance between the azimuth axis and point G
$Z_G$ = distance between azimuth axis and elevation axis
$Z_P$ = distance between point G and point P
$\alpha$ = rotation angle around azimuth axis
$\beta$ = rotation angle around elevation axis

## Introduction

IN multibeam satellite communication systems, it is necessary for the beam pointing direction of the onboard antennas to be accurately controlled because of their narrow beamwidth. An antenna drive control system (ADCS), which drives the antenna's reflector, has been developed to satisfy the pointing accuracy requirement.[1,2] Since this onboard control system operates in a space orbit, it is important to carry out ground tests under a simulated space environment in order to precisely verify the control characteristics. However, the trend toward large antennas makes it quite difficult to simulate the space

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